NETWORK RESPONSE TO A FREQUENCY PULSE MODULATED WAVE

by
GIORGIO BARZILAI

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Microwave Research Institute
Polytechnic Institute of Brooklyn
55 Johnson Street
Brooklyn 1, New York

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SUMMARY

The present report is concerned with a theoretical study of the response of a linear two terminal network to a frequency pulse modulated wave, if this network is followed by an ideal discriminator. We have assumed the modulating function to be of the form

$$S(t) = \frac{1}{1 + (\mu t)^2}$$

where t is the time variable, and μ a parameter characterizing the "duration of the pulse".

In Section I the response $r(t)$ of a general two terminal network is found for the special case when $\Delta \omega_o/2\mu$ is an integer, where $\Delta \omega_o$ is the deviation from the carrier frequency ω_o.

In Section II the output of the discriminator $r_d(t)$ is determined, under the assumption that $\Delta \omega_o/2\mu = 1$. It is shown that such hypothesis corresponds to cases of practical interest.

Section III is concerned with expansions of the exponential integral of complex argument. This function enters in the expressions of $r(t)$ and $r_d(t)$.

In Section IV the formulae obtained in Section II are specialized for the case of a single parallel RLC circuit tuned on the carrier frequency ω_o. It is assumed that $2\omega_o >> \mu$ and $2\omega_o >> w$, where $2w$ is the total bandwidth of the circuit between half power points.

In Section V asymptotic expressions are derived for the cases of $w/\mu \longrightarrow \infty$ and $w/\mu \longrightarrow 0$. For $w/\mu \longrightarrow \infty$ it is found that time dependence of $v_d(t)$ approaches $S(t)$ as, of course, it should be. In such a case, the parallel tuned circuit degenerates in to a pure resistance ($w = \infty$), if it is assumed $\mu \neq 0$. In deriving these asymptotic expressions the assumptions previously made are of course maintained. For instance, it is always assumed $2\omega_o >> w$.

Section VI is concerned with numerical computations, based on the formulae obtained in Section IV. Calculations have been carried out for $-4 \leq \mu t \leq 4$, and for $w/\mu \longrightarrow \infty$, 10, 4, 2, 1, 0.4, 0.1, 0.01$\longrightarrow$0. These calculations show that the time dependence of $v_d(t)$ is very similar to that of the modulating function $S(t)$, provided $w/\mu > 1$. If $w/\mu < 1$ the input signal becomes very distorted. We can say that for $\infty > w/\mu > 1$ the output is similar to the input, i.e. $1/1+(\mu t)^2$. For $0.1 < w/\mu < 1$ the output is very critically dependent on time, and therefore difficult to calculate. In the interval $0 < w/\mu < 0.1$ the output does not vary too much and it is similar to that given by the asymptotic formula for $w/\mu \longrightarrow 0$, i.e. $\mu t/1+(\mu t)^2$.

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Introduction

Transients solutions for linear networks, subject to a frequency modulated driving function, have been obtained for a few typical cases. Salinger[1] considers the response of an ideal band pass filter followed by an ideal discriminator, to a frequency "step". Gold[2] extended Salinger's solution for the case of a frequency "square pulse". Hok[3] and Ekstein and Schiffman[4] have investigated the response of a linear network, without discriminator, to a linearly increasing frequency.

The solutions recalled were possible in virtue of the simplicity of the modulating function chosen. Salinger picked the simplest type of modulating function, i.e. a constant (in Heaviside's sense). Hok considered the next function in order of complexity, i.e. the linear function.

It seems therefore natural to ask: Is there some other type of simple modulating function which, besides being of practical interest, will lead to integrations which can be handled? If one considers the formal expression for the response of a general linear network subject to a frequency modulated driving function, it is easy to see that modulating functions of the form

$$S(t) = \frac{1}{1 + \mu^2 t^2}$$

where t represents time and μ a parameter, lead to expressions which can be integrated, at least for values of the frequency swing integral multiples of 2μ. The function $S(t)$ is also suitable to represent a "pulse", which is a type of modulation of practical importance. We have therefore chosen this function for our investigation.

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1 H. Salinger, "Transients in Frequency Modulation", Proc. I.R.E., August, 1942.

2 B. Gold, "Transients in Frequency Modulation", Report R-154-47, PIB-103, Microwave Research Institute of the Polytechnic Institute of Brooklyn, 1947.

3 G. Hok, "Response of Linear Resonant Systems to Excitation of a Frequency Varying Linearly with Time", Jour. App. Phys., March 1948.

4 H. Ekstein and T. Schiffman, "Response of a Linear Network with a Linearly Variable Frequency as obtained in Sweep Frequency Test", Proceedings of the National Electronics Conference, Vol. VII, 1951.

Section I - Response of a Two Terminal Network

The response of a two terminal network can be expressed, by means of the superposition integral, as follows

$$r(t) = \int_{-\infty}^{t} f(\tau)\, g(t - \tau)\, d\tau \tag{1}$$

where:

$r(t)$ is response

$f(\tau)$ is driving function

$g(t)$ is the Green's function, i.e. the response of the network in question to a unit impulse. The lower limit of the integral has been assumed to be $-\infty$ since the function $f(\tau)$ that we shall consider starts at $t = -\infty$.

For a non-impulsive network the Laplace transform of $g(t)$, that we shall indicate with $G(p)$, is a proper rational fraction, and can be expanded as follows:

$$G(p) = \sum_{s=1}^{n} \frac{A_s}{p - p_s} \tag{2}$$

where p_s are the roots of the denominator of $G(p)$, which are assumed to be all distinct. We now assume a complex driving function of the form

$$\bar{f}(t) = F_o\, e^{j\theta(t)} = F_o\, e^{j(\omega_o t + \Delta\omega_o \int_{-\infty}^{t} S(t)\, dt)} \tag{3}$$

The instantaneous frequency, is defined uniquely, since F_o is constant with time, as

$$\frac{d\theta}{dt} = \omega_o + \Delta\omega_o\, S(t) \tag{4}$$

If now we assume

$$S(t) = \frac{1}{1 + (\mu t)^2} \tag{5}$$

the frequency of the driving function will vary in the pulse like manner depicted in Fig. 1.

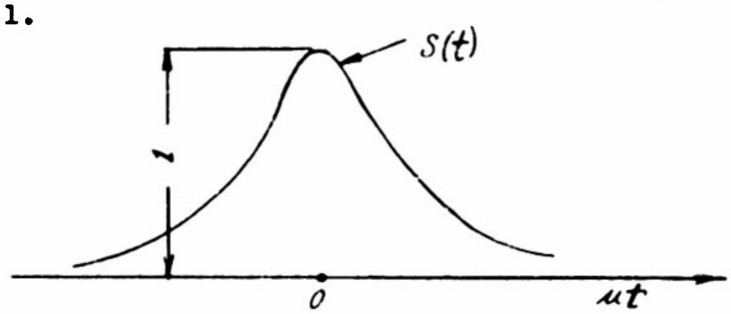


Fig. 1

Since $\int_{-\infty}^{t} S(t)\,dt = \frac{1}{\mu}(\tan^{-1}\mu t + \frac{\pi}{2})$ Eq. (3) can be written:

$$\bar{f}(t) = F_o\, e^{j\left[\omega_o t + \frac{\Delta\omega_o}{\mu}(\tan^{-1}\mu t + \frac{\pi}{2})\right]} \tag{6}$$

If now the function (3) is applied to the two terminals of the system (2), the typical s component of the response can be written, by using the superposition integral, as follows

$$\bar{r}_s = A_s\, e^{p_s t} \int_{-\infty}^{t} e^{-p_s \tau}\, \bar{f}(\tau)\, d\tau$$

and using Eq. (6)

$$\bar{r}_s = \underline{F_o}\, A_s\, e^{p_s t} \int_{-\infty}^{t} e^{+\Gamma_s \tau + j\frac{\Delta\omega_o}{\mu}\tan^{-1}\mu\tau}\, d\tau \tag{7}$$

Where we have let

$$\Gamma_s = j\,\omega_o - p_s \qquad\qquad \underline{F_o} = F_o\, e^{j\frac{\Delta\omega_o}{\mu}\frac{\pi}{2}} \tag{8}$$

if now we recall that

$$\tan^{-1} z = \frac{1}{2j}\log\frac{1 + jz}{1 - jz}$$

Eq. (7) becomes

$$\bar{r}_s = \underline{F_o}\, A_s\, e^{p_s t} \int_{-\infty}^{t} e^{\Gamma_s \tau} \left(\frac{1 + j\mu\tau}{1 - j\mu\tau}\right)^{\frac{\Delta\omega_o}{2\mu}} d\tau \tag{9}$$

The integral appearing in Eq. (9) can be easily evaluated if $\Delta\omega_o/2\mu$ is an integer. If we let $\Delta\omega_o/2\mu = q$, we are concerned with the following integral

$$\int_{-\infty}^{t} \left(\frac{1 + j\mu t}{1 - j\mu t}\right)^{q} e^{\Gamma_s t}\, dt \tag{10}$$

To transform this integral we notice that, in general, we can write

$$\frac{(z + 1)^{q-1}}{(z - 1)^q} = \frac{N_1}{z - 1} + \frac{N_2}{(z - 1)^2} + \dots + \frac{N_q}{(z - 1)^q} \tag{11}$$

where

$$N_r = \frac{1}{(q-r)!}\ \frac{d^{q-r}}{d\,Z^{q-r}}(Z+1)^{q-1}\Big|_{Z=1} \qquad r = 1, 2, \ldots q \tag{12}$$

since the function on the left side of Eq. (11) has a pole of order q at Z=1.

Performing the differentiations indicated in Eq. (12) we get

$$N_r = \binom{q-1}{q-r} 2^{r-1}$$

Eq.(10) therefore becomes

$$\int_{-\infty}^{t} \left(\frac{1+j\mu t}{1-j\mu t}\right)^q e^{\Gamma_s t}\, dt = (-1)^q \int_{-\infty}^{t} \left(\frac{j\mu t+1}{j\mu t-1}\right)^q e^{\Gamma_s t}\, dt$$

$$= (-1)^q \int_{-\infty}^{t} \left[\frac{t-\frac{j}{\mu}}{t+\frac{j}{\mu}} + N_2 \frac{1}{j\mu} \frac{t-\frac{j}{\mu}}{(t+\frac{j}{\mu})^2} + N_3 \frac{1}{(j\mu)^2} \frac{t-\frac{j}{\mu}}{(\frac{j}{\mu}+t)^3} + \ldots\right.$$

$$\left.\ldots N_q \frac{1}{(j\mu)^{q-1}} \frac{t-\frac{j}{\mu}}{(\frac{j}{\mu}+t)^q}\right] e^{\Gamma_s t}\, dt$$

Letting now

$$Z = t + \frac{j}{\mu}; \quad Z - \frac{2j}{\mu} = t - \frac{j}{\mu} \qquad t = Z - \frac{j}{\mu}, \quad dt = dZ$$

we get

$$\int_{-\infty}^{t} \left(\frac{1+j\mu t}{1-j\mu t}\right)^q e^{\Gamma_s t}\, dt = (-1)^q e^{-\frac{j}{\mu}\Gamma_s} \int_{-\infty}^{t+j\frac{1}{\mu}} \left[N_1 + \frac{2N_1+N_2}{j\mu}\frac{1}{Z}\right.$$

$$\left. + \frac{2N_2+N_3}{(j\mu)^2}\frac{1}{Z^2} + \ldots + \frac{2N_{q-1}+N_q}{(j\mu)^{q-1}}\frac{1}{Z^{q-1}} + \frac{2N_q}{(j\mu)^q}\frac{1}{Z^q}\right] e^{\Gamma_s Z}\, dZ$$

(13)

where the path of integration in the complex Z plan is indicated in Fig. 2.

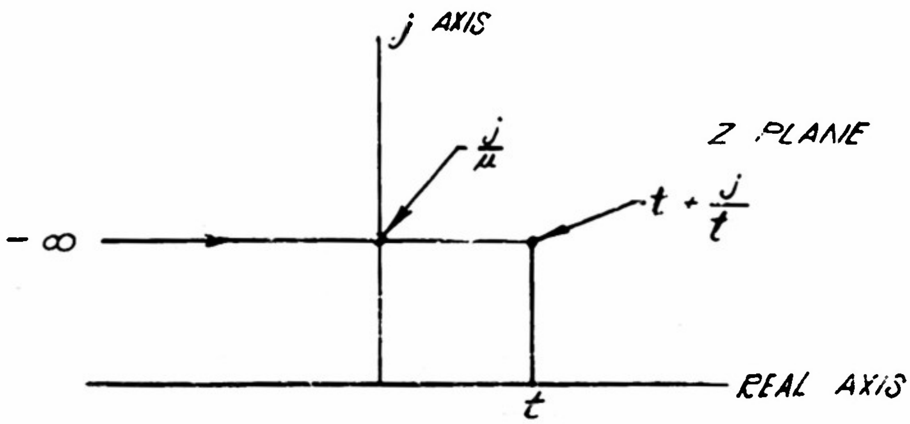


Fig. 2

All the integrals appearing on the right hand side of Eq. (13) can be expressed in terms of the exponential integral

$$I_s = I_{r,s} + j\, I_{i,s} = \int_{-\infty}^{(j/\mu + t)} \frac{e^{\Gamma_s Z}}{Z}\, dZ \qquad (14)$$

this is apparent if we recall that in general

$$\int \frac{e^{\Gamma_s Z}}{Z^m}\, dZ = - e^{\Gamma_s Z} \left(\frac{1}{m-1} \frac{1}{Z^{m-1}} + \frac{\Gamma_s}{(m-1)(m-2)} \frac{1}{Z^{m-2}} + \dots + \frac{\Gamma_s^{\,m-2}}{(m-1)!} \frac{1}{Z} \right)$$

$$+ \frac{\Gamma_s^{\,m-1}}{(m-1)!} \int \frac{e^{\Gamma_s Z}}{Z}\, dZ \qquad m > 1 \qquad (15)$$

Eq. (15) is obtained by successive integration by parts. In applying this formula to Eq. (13) we have to recall that

$$R_e \left[\Gamma_s \right] = R_e \left[- p_s \right] > 0 \quad .$$

and therefore

$$e^{\Gamma_s Z} \longrightarrow 0 \qquad \text{when} \qquad R_e \left[Z \right] \longrightarrow -\infty \quad .$$

By inserting (15) into (13) we finally attain the following expression for (9)

$$\bar{r}_s = (-1)^q \underline{F_o} A_s e^{p_s t - \frac{j}{\mu}\Gamma_s} \Bigg\{ \Bigg[\frac{2N_1 + N_2}{j\mu} + \frac{2N_2 + N_3}{(j\mu)^2} \Gamma_s$$

$$+ \cdots + \frac{2N_q}{(j\mu)^q} \frac{\Gamma_s^{q-1}}{(q-1)!} \Bigg] I_s$$

$$+ e^{\Gamma_s (t + \frac{j}{\mu})} \Bigg[\frac{1}{\Gamma_s} - \frac{2N_2 + N_3}{(j\mu)^2} \frac{1}{t + \frac{j}{\mu}} - \frac{2N_3 + N_4}{(j\mu)^3} \Big(\frac{1}{2} \frac{1}{(t + \frac{j}{\mu})^2}$$

$$+ \frac{\Gamma_s}{2} \frac{1}{(t + \frac{j}{\mu})} \Big) - \cdots - \frac{2N_q}{(j\mu)^q} \Big(\frac{1}{q-1} \frac{1}{(t + \frac{j}{\mu})^{q-1}} + \frac{\Gamma_s}{(q-1)(q-2)} \frac{1}{(t + \frac{j}{\mu})^{q-2}}$$

$$+ \cdots + \frac{\Gamma_s^{q-2}}{(q-1)!} \frac{1}{t + \frac{j}{\mu}} \Big) \Bigg] \Bigg\} \qquad (16)$$

Equation (16) shows that the response of a general network to a frequency modulated driving function, whose modulating function is of the form (5), can be expressed in terms of exponential integrals of complex argument, provided the ratio $\Delta\omega_o/2\mu$ is a positive integer.

Section II - Output of the Limiter-Discriminator

Equation (16) gives the sth component of the response. What is usually required, however, is the output of an ideal limiter-discriminator network, to which the response of the system is applied. For ideal limiter-discriminator we mean a device, which first reduces the amplitude of the applied signal to a value constant with time (limiter), and whose output is proportional to the time derivative of the phase of this constant amplitude signal (discriminator).

Formally it is a simple matter to find such derivative, but to carry out the actual calculations is usually very cumbersome.

In general, if we write the response as

$$\bar{r}(t) = \sum_{s=1}^{n} \bar{r}_s = \sum_{s=1}^{n} (R_s + j X_s) = R + j X \qquad (17)$$

its phase angle can be written

$$\psi = \tan^{-1} \frac{X}{R} \qquad (18)$$

and therefore

$$K \frac{d\psi}{dt} = K \frac{R X' - X R'}{R^2 + X^2} = r_d(t) \qquad (19)$$

where K is a constant which is dependent upon the characteristics of the discriminator.

From the preceding section it is apparent that to derive a general expression for the response r_d for any integer q is a very complicated matter. In what follows we shall therefore only consider cases where

$$q = \frac{\Delta \omega_0}{2 \mu} = 1 \qquad (20)$$

That such an assumption refers to cases which can be of a practical importance can be easily shown.

From Eq. (5) it is evident that we have to define the duration of the pulse. A reasonable definition for this quantity can be the following: The duration of the pulse is defined as twice the time T necessary to reduce S(t) from 1 to 1/10. Using this definition we can relate μ with the duration of the pulse 2T. We have

$$0.1 = \frac{1}{1 + (\mu T)^2}$$

or

$$\mu = \frac{3}{T} \qquad (21)$$

For a pulse duration of 1 μ sec ($2T = 10^{-6}$ sec), we get from Eq. (21)

$$\mu = 6.10^6$$

and taking into account Eq. (20)

$$\Delta\omega_o = 12.10^6 \text{ rad/sec.}$$

In other words with a pulse duration of 1 μ sec, to fulfill condition (20) the frequency deviation must be about 1.9 Mc/sec.

Assumption (20) simplifies very much expression (16), and we have

$$\bar{r}_s = -A_s F_o e^{p_s t - \frac{j}{\mu}\Gamma_s} \left[2\frac{j}{\mu} I_s - \frac{e^{\Gamma_s\left(\frac{j}{\mu}+t\right)}}{\Gamma_s} \right] \qquad (22)$$

since according to the second of (8) $\underline{F_o} = F_o e^{j\pi} = -F_o$. If now we let

$$p_s = \alpha_s + j\beta_s \qquad , \qquad A_s = |A_s| e^{j\phi_s} \qquad (23)$$

and recall the first of (8) and (14), we get

$$\bar{r}_s = -F_o|A_s| \left[2(I_{r,s} + j I_{i,s}) \frac{j}{\mu} e^{\alpha_s t + \frac{\omega_o - \beta_s}{\mu} + j\left(\beta_s t + \phi_s + \frac{\alpha_s}{\mu}\right)} \right.$$

$$\left. + \frac{\alpha_s + j(\omega_o - \beta_s)}{\alpha_s^2 + (\omega_o - \beta_s)^2} e^{j(\phi_s + \omega_o t)} \right] \qquad (24)$$

The second term of this equation is easily recognized to be the steady state. Separating the real and imaginary part of the right hand side of Eq. (24) we get

$$R_s = F_o|A_s| \left\{ 2\frac{e}{\mu}^{\alpha_s t + \frac{\omega_o - \beta_s}{\mu}} \left[I_{i,s} \cos\gamma_s + I_{r,s} \sin\gamma_s \right] \right.$$

$$\left. + \frac{1}{\alpha_s^2 + (\omega_o - \beta_s)^2} \left[(\omega_o - \beta_s)\sin\delta_s - \alpha_s \cos\delta_s \right] \right\} \qquad (25)$$

$$X_s = F_o|A_s| \left\{ 2\frac{e}{\mu}^{-\alpha_s t} + \frac{\omega_o - \beta_s}{\mu} \left[-I_{r,s} \cos \gamma_s + I_{i,s} \sin \gamma_s \right] \right.$$

$$\left. - \frac{1}{\alpha_s^2 + (\omega_o - \beta_s)^2} \left[(\omega_o - \beta_s) \cos \delta_s + \alpha_s \sin \delta_s \right] \right. \tag{26}$$

where

$$\gamma_s = \beta_s t + \phi_s + \frac{\alpha_s}{\mu} \quad \text{and} \quad \delta_s = \phi_s + \omega_o t \tag{27}$$

the derivatives of R_s and X_s are as follows:

$$R_s' = F_o|A_s| 2 \frac{e^{\frac{\omega_o - \beta_s}{\mu}}}{\mu} \left\{ e^{-\alpha_s t} \left[-\beta_s I_{i,s} \sin \gamma_s + I_{i,s}' \cos \gamma_s + \beta_s I_{r,s} \cos \gamma_s + I_{r,s}' \sin \gamma_s \right] \right.$$

$$\left. + \alpha_s e^{-\alpha_s t} \left[I_{i,s} \cos \gamma_s + I_{r,s} \sin \gamma_s \right] \right\} + \frac{F_o |A_s|}{\alpha_s^2 + (\omega_o - \beta_s)^2} \left[\alpha_s \omega_o \sin \delta_s + \omega_o (\omega_o - \beta_s) \cos \delta_s \right] \tag{28}$$

$$X_s' = F_o|A_s| 2 \frac{e^{\frac{\omega_o - \beta_s}{\mu}}}{\mu} \left\{ e^{-\alpha_s t} \left[\beta_s I_{r,s} \sin \gamma_s - I_{r,s}' \cos \gamma_s + \beta_s I_{i,s} \cos \gamma_s + I_{i,s}' \sin \gamma_s \right] \right.$$

$$+ \alpha_s e^{-\alpha_s t} \left[-I_{r,s} \cos \gamma_s + I_{i,s} \sin \gamma_s \right] + \frac{F_o |A_s|}{\alpha_s^2 + (\omega_o - \beta_s)^2} \left[\omega_o (\omega_o - \beta_s) \sin \delta_s - \alpha_s \omega_o \cos \delta_s \right] \tag{29}$$

Formulae (25), (26), (28), (29), together with (19), allow the calculation of the output of the limiter-discriminator, when the network in question is driven by the function (ϵ), and $\Delta \omega_o / 2\mu = 1$.

Section III - Expansions for the Integral (14)

The integral (14) can be easily transformed into a standard form as follows:

$$I = \int_{-\infty + \frac{j}{\mu}}^{j/\mu + t} \frac{e^{tZ}}{Z} dZ = \int_{(+\infty - \frac{j}{\mu})\Gamma}^{\lambda} \frac{e^{-Z}}{Z} dZ \tag{30}$$

where

$$\lambda = -\Gamma\left(\frac{j}{\mu} + t\right) = \left[\alpha - j(\omega_0 - \beta)\right]\left(\frac{j}{\mu} + t\right) \tag{31}$$

$$= M + j N = \rho e^{i\theta}$$

$$M = \alpha t + \frac{\omega_0 - \beta}{\mu} \qquad N = \frac{\alpha}{\mu} - (\omega_0 - \beta) t \tag{32}$$

Now we have

$$I = C + \log \lambda - \int_0^{\lambda} \frac{1 - e^{-Z}}{Z} dZ = C + \log \lambda - \sum_{m=1}^{\infty} (-1)^{m-1} \frac{\lambda^m}{m\, m!} \tag{33}$$

where $C = 0.5772$ is the Euler-Mascheroni constant.

Since the path of integration does not encircle the origin (Fig. 2)

$$\log \lambda = \log \rho + j\,\theta \tag{34}$$

we can therefore write

$$I = I_r + j I_i = C + \log \rho + \sum_{m=1}^{\infty} (-1)^m \frac{\rho^m}{m\, m!} \cos m\theta + i\left[\theta + \sum_{m=1}^{\infty} (-1)^m \frac{\rho^m}{m\, m!} \sin m\theta\right] \tag{35}$$

A useful asymptotic expansion for the integral (30) is easily obtained by successive integration by parts, and it is the following

$$I \sim -\frac{e^{-\lambda}}{\lambda}\left(1 - \frac{1!}{\lambda} + \frac{2!}{\lambda^2} - \frac{3!}{\lambda^3} + \dots\right) \tag{36}$$

It is easy to derive an expression for I' which appears in Eq. (28) and Eq. (29)

$$I' = \frac{d\,I}{dt} = \frac{d\,I}{d\lambda}\;\frac{d\lambda}{dt} = \frac{e^{-\lambda}}{\lambda}\left[\alpha - j(\omega_o - \beta)\right] = \frac{\alpha - j\,(\omega_o - \beta)}{\alpha t + \dfrac{\omega_o - \beta}{\mu} + j\left[\dfrac{\alpha}{\mu} - (\omega_o - \beta)t\right]}\; e^{-\lambda} \tag{37}$$

Section IV - Single Tuned Circuit

We want now to apply the formula of the preceding section to calculate the output of the discriminator, for the case sketched in Fig. 3.

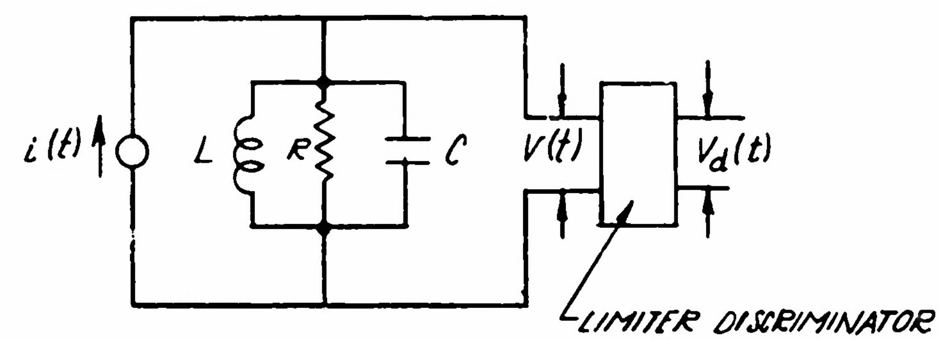


Fig. 3

We shall assume that the circuit is tuned on the carrier frequency $\omega_o = \frac{1}{\sqrt{LC}}$, and

$$\frac{2\,\omega_o}{w} >> 1 \tag{38}$$

where 2w is the total bandwidth between half power points. The current $\bar{i}(t)$ is the complex driving function, and its expression, if we assume $F_o = 1$ and $\Delta\,\omega_o/2\mu = 1$, follows from (6)

$$\bar{I}\,(t) = e^{j\left[\omega_o\, t + 2\,\tan^{-1}\mu t + \frac{\pi}{2}\right]} \tag{39}$$

From (2) we have

$$G(p) = \frac{1}{2C} \frac{1}{j\sqrt{\omega_o^2 - \frac{1}{4R^2C^2}}} \left(\frac{p_1}{p - p_1} - \frac{p_2}{p - p_2}\right) \tag{40}$$

where

$$\begin{matrix} p_1 \\ p_2 \end{matrix} = -\frac{1}{2RC} \pm j\sqrt{\omega_o^2 - \frac{1}{4R^2C^2}} \quad , \quad \omega_o = \frac{1}{\sqrt{LC}} \tag{41}$$

Introducing now the Q of the circuit, we have

$$Q = \frac{R}{\omega_o L} = \omega_o RC \qquad w = \frac{\omega_o}{2Q} = \frac{1}{2RC}$$

$$\begin{matrix} p_1 \\ p_2 \end{matrix} \approx -w \pm j\omega_o \qquad G(p) \approx \frac{1}{2C}\left(\frac{1}{p - p_1} + \frac{1}{p - p_2}\right) \tag{42}$$

With reference to formulae (25), (26) and (27), we recognize that

$$|A_1| = |A_2| = \frac{1}{2C} \quad , \quad \alpha_1 = \alpha_2 = -w \quad , \quad \begin{matrix} \beta_1 \\ \beta_2 \end{matrix} = \pm\omega_o$$

$$\phi_1 = \phi_2 = 0 \quad , \quad \begin{matrix} \gamma_1 \\ \gamma_2 \end{matrix} = \pm\omega_o t - \frac{w}{\mu} \quad , \quad \begin{matrix} \delta_1 \\ \delta_2 \end{matrix} = \pm\omega_o t \tag{43}$$

Eq. (17), taking into account (24), (42), (43), becomes

$$\bar{v}(t) = R + jX = \sum_{s=1}^{2} \bar{v}_s(t) = -\frac{1}{2C}\left[2 I_1(\lambda_1)\frac{j}{\mu} e^{-wt + j\gamma_1} - \frac{e^{j\delta_1}}{w}\right.$$

$$\left. + 2 I_2(\lambda_2)\frac{j}{\mu} e^{-wt + \frac{2\omega_o}{\mu} + j\gamma_2} + \frac{e^{j\delta_2}}{2\omega_o}\right] \tag{44}$$

where, recalling Eqs. (31) and (32),

$$\lambda_1 = -(wt + j\frac{w}{\mu}) \qquad \lambda_2 = \frac{2\omega_o}{\mu} - wt - j\,(2\omega_o t + \frac{w}{\mu}) \tag{45}$$

We now want to investigate the magnitude of the term containing $I_2(\lambda_2)$ in Eq. (44), and compare it with the magnitude of the term containing $I_1^2(\lambda_1^2)$, in order to see if some simplication is possible.

In the following we shall assume that

$$\frac{2\,\omega_o}{\mu} >> 1 \tag{46}$$

Let's start investigating the term containing $I_2(\lambda_2)$. We have

$$|\lambda_2| = \left[(\frac{2\omega_o}{\mu} - wt)^2 + (2\omega_o t + \frac{w}{\mu})^2\right]^{1/2}$$

$$= \left[(\frac{2\omega_o}{\mu})^2 + (wt)^2 + (2\omega_o t)^2 + (\frac{w}{\mu})^2\right]^{1/2} \geq \frac{2\omega_o}{\mu} \tag{47}$$

Comparing (47) and (46) we can conclude that $|\lambda_2| >> 1$. From Eq. (36) we get, for the magnitude of the term containing $I_2(\lambda_2)$,

$$\left|2I_2(\lambda_2)\,\frac{1}{\mu}\,e^{-wt+\frac{2\omega_o}{\mu}+j\gamma_2}\right| \approx \left|-\frac{2}{\mu}\,\frac{1}{\frac{2\omega_o}{\mu} - wt - j(2\omega_o t + \frac{w}{\mu})}\right| \leq \frac{1}{\omega_o} \tag{48}$$

Let's now investigate the term, containing $I_1(\lambda_1)$. We have, letting $Z = \xi - j\frac{w}{\mu}$,

$$\left|2I_1(\lambda_1)\,\frac{1}{\mu}\,e^{-wt + j\gamma_1}\right| = \frac{2}{\mu}\,e^{-wt}\left|\int_{+\infty - j\frac{w}{\mu}}^{-wt - j\frac{w}{\mu}} \frac{e^{-Z}}{Z}\,dZ\right|$$

$$= \frac{2}{\mu}\,e^{-wt}\left|\int_{+\infty}^{-wt} \frac{e^{-(\xi - j\frac{w}{\mu})}}{\xi - j\frac{w}{\mu}}\,d\xi\right| \leq \frac{2}{\mu}\,e^{-wt}\int_{+\infty}^{-wt}\left|\frac{e^{-(\xi - j\frac{w}{\mu})}}{\xi - j\frac{w}{\mu}}\,d\xi\right|$$

$$= -\frac{2}{\mu}\,e^{-wt}\int_{+\infty}^{-wt}\frac{e^{-\xi}}{\sqrt{\xi^2 + (\frac{w}{\mu})^2}}\,d\xi \leq -\frac{2}{w}\,e^{-wt}\int_{+\infty}^{-wt} e^{-\xi}\,d\xi = \frac{2}{w} \tag{49}$$

Since in virtue of (38)

$$\frac{1}{\omega_0} << \frac{2}{v}$$

we can conclude that, in Eq. (44), the term containing $I_2(\lambda_2)$ can be neglected in comparison with the term containing $I_1(\lambda_1)$. With this simplification, and keeping into account (38), we easily obtain the following expressions for the real and imaginary part of $\bar{v}(t)$:

$$R = \frac{1}{2C}\left\{\left[\frac{2}{\mu} e^{-vt} \left(I_{i,1} \cos\frac{v}{\mu} - I_{r,1} \sin\frac{v}{\mu}\right) + \frac{1}{v}\right] \cos \omega_0 t + \frac{2}{\mu} e^{-vt} \left(I_{i,1} \sin\frac{v}{\mu} + I_{r,1} \cos\frac{v}{\mu}\right) \sin \omega_0 t\right\} \tag{50}$$

$$X = \frac{1}{2C}\left\{\frac{2}{\mu} e^{-vt} \left(- I_{r,1} \cos\frac{v}{\mu} - I_{i,1} \sin\frac{v}{\mu}\right) \cos \omega_0 t + \left[\frac{2}{\mu} e^{-vt} \left(-I_{r,1} \sin\frac{v}{\mu} + I_{i,1} \cos\frac{v}{\mu}\right) + \frac{1}{v}\right] \sin \omega_0 t\right\} \tag{51}$$

or letting

$$A = \frac{1}{2C}\left[\frac{2}{\mu} e^{-vt} \left(I_{i,1} \cos\frac{v}{\mu} - I_{r,1} \sin\frac{v}{\mu}\right) + \frac{1}{v}\right] \tag{52}$$

$$B = \frac{1}{2C} \frac{2}{\mu} e^{-vt} \left(I_{i,1} \sin\frac{v}{\mu} + I_{r,1} \cos\frac{v}{\mu}\right) \tag{53}$$

we can write

$$R = A \cos \omega_0 t + B \sin \omega_0 t \tag{54}$$

$$X = -B \cos \omega_0 t + A \sin \omega_0 t \tag{55}$$

or

$$R = \sqrt{A^2 + B^2} \cos(\omega_0 t - \psi) \qquad \psi = \tan^{-1}\frac{B}{A} \tag{56}$$

$$X = \sqrt{A^2 + B^2} \cos\left(\omega_0 t + \frac{\pi}{2} - \psi\right) \tag{57}$$

Except for the constant $K \omega_0$ we can therefore write

$$v_d = - K \frac{d\psi}{dt} \tag{58}$$

or

$$v_d(t) = -\frac{A_1B' - B_1A'}{A_1^2 + B_1^2}\, 2\, w\, C\, K \tag{59}$$

$$A_1 = 2C\, w\, A = 2\frac{w}{\mu}\, e^{-\frac{w}{\mu}u}\left[I_i(\lambda)\cos\frac{w}{\mu} - I_r(\lambda)\sin\frac{w}{\mu}\right] + 1 \tag{60}$$

$$B_1 = 2C\, w\, B = 2\frac{w}{\mu}\, e^{-\frac{w}{\mu}u}\left[I_i(\lambda)\sin\frac{w}{\mu} + I_r(\lambda)\cos\frac{w}{\mu}\right] \tag{61}$$

and taking into account the first of Eq. (45)

$$2\, C\, A' = -A_1 + 1 - \frac{2}{1+u^2} \tag{62}$$

$$2\, C\, B' = -B_1 + 2\frac{u}{1+u^2} \tag{63}$$

where we have

$$u = \mu\, t \tag{64}$$

Section V - Asymptotic Formulae

It is now interesting to investigate the form that Eq. (59) takes when $w \longrightarrow \infty$ or $w \longrightarrow 0$. The first case obviously corresponds to the case of a pure resistance across the current generator (see Fig. 3). We expect therefore to find v_d proportional to the modulating function (5). The second case corresponds to a lossless parallel tuned circuit across the current generator. However, in this case the form of v_d is not so obvious as in the first case.

To find the asymptotic formula for v_d when $w \longrightarrow \infty$ we use the asymptotic expression (36). In order to get a remainder for the quantities A_1, B_1, A', B' that vanishes at least as μ/w, we have to take two (or more) terms of the expansion (36). Proceeding in this way we get

$$A_1 = - \frac{2}{1 + u^2} + 1 - 4 \frac{\mu}{w} \frac{u}{(1 + u^2)^2} + 0 \left(\frac{\mu}{w^2}\right) \tag{65}$$

$$B_1 = \frac{2u}{1 + u^2} - 2 \frac{\mu}{w} \frac{1 - u^2}{(1 + u^2)^2} + 0 \left(\frac{\mu}{w^2}\right) \tag{66}$$

$$2\,C\,A' = + 4 \frac{\mu}{w} \frac{u}{(1 + u^2)^2} + 0 \left(\frac{\mu}{w^2}\right) \tag{67}$$

$$2\,C\,B' = + 2 \frac{\mu}{w} \frac{1 - u^2}{(1 + u^2)^2} + 0 \left(\frac{\mu}{w^2}\right) \tag{68}$$

introducing these expressions in Eq. (59), and keeping only the term of lowest order in μ/w, we get

$$\lim_{w \to \infty} v_d(t) = 2\mu \frac{1}{1 + u^2} \tag{69}$$

the right hand side of Eq. (69) is simply $d/dt\,(2 \tan \mu t)$ i.e. the modulating function $S(t)$. It should be noted that the asymptotic expression (69) could have also been obtained by introducing the first term of Eqs. (36) in (60) and (61) and then evaluating $C\,A'$ and $C\,B'$. Such a procedure is equivalent to differentiating the asymptotic expansion (36). In this case it is easy to see that one term only of the asymptotic expression leads to a remainder which vanishes at least as μ/w. To find the asymptotic formula for $w/\mu \to 0$, we use the series expansion (33). Proceeding in this way, and keeping only the lowest order in w, we get

$$\lim_{w \to \infty} v_d = - 2wK \frac{u}{1 + u^2} = v_{d,\infty}(t) \tag{70}$$

Section VI - Numerical Computations

Using Eqs. (60), (61), (62), (63) and (59) numerical computations have been carried out for $-4 \leq u = \mu t \leq 4$ for the following values of $w/\mu = \to \infty$, 10, 4, 2, 1, 04, 0.1, 0.01, $\to 0$. The results of these calculations have been recorded in Figs. MRI-13106, MRI-13107, MRI-13108, MRI-13109, MRI-13110, MRI-13111, MRI-13112, MRI-13113, MRI-13114 and MRI-13115, where v_d' is plotted against u, and we have let $v_d' = v_d/v_{d,\infty}(0)$. $v_{d,\infty}(0) = 2\mu K$ as it is apparent from Eq. (70).

The maximum positive value of v_d' has been normalised to 1. To get the actual value of v_d', i. e. the value (59) divided by $2\mu K$, the ordinates of the diagram have to be multiplied by a constant H whose value is indicated in each diagram.

The numerical evaluation of the integral (30) has been carried out by means of the expansions (35) or (36). When possible the values tabulated by the Computation Laboratory, 150 Nassau Street, New York, N.Y. have been used. However, the available Tables are, at present, rather limited, and in particular the values corresponding to negative real part of the independent variable, have not been tabulated.

The results of the computation for $I_o(\lambda)$ and $I_1(\lambda)$ have been recorded in Figs. MRI-13116, MRI-13117, MRI-13118, MRI-13119, MRI-131120 and MRI-13121. In these diagrams the points marked with an arrow refer to the values tabulated by the National Laboratory. The other values have been computed by us.

In the computation we have let

$$\lambda = x + j\,y$$

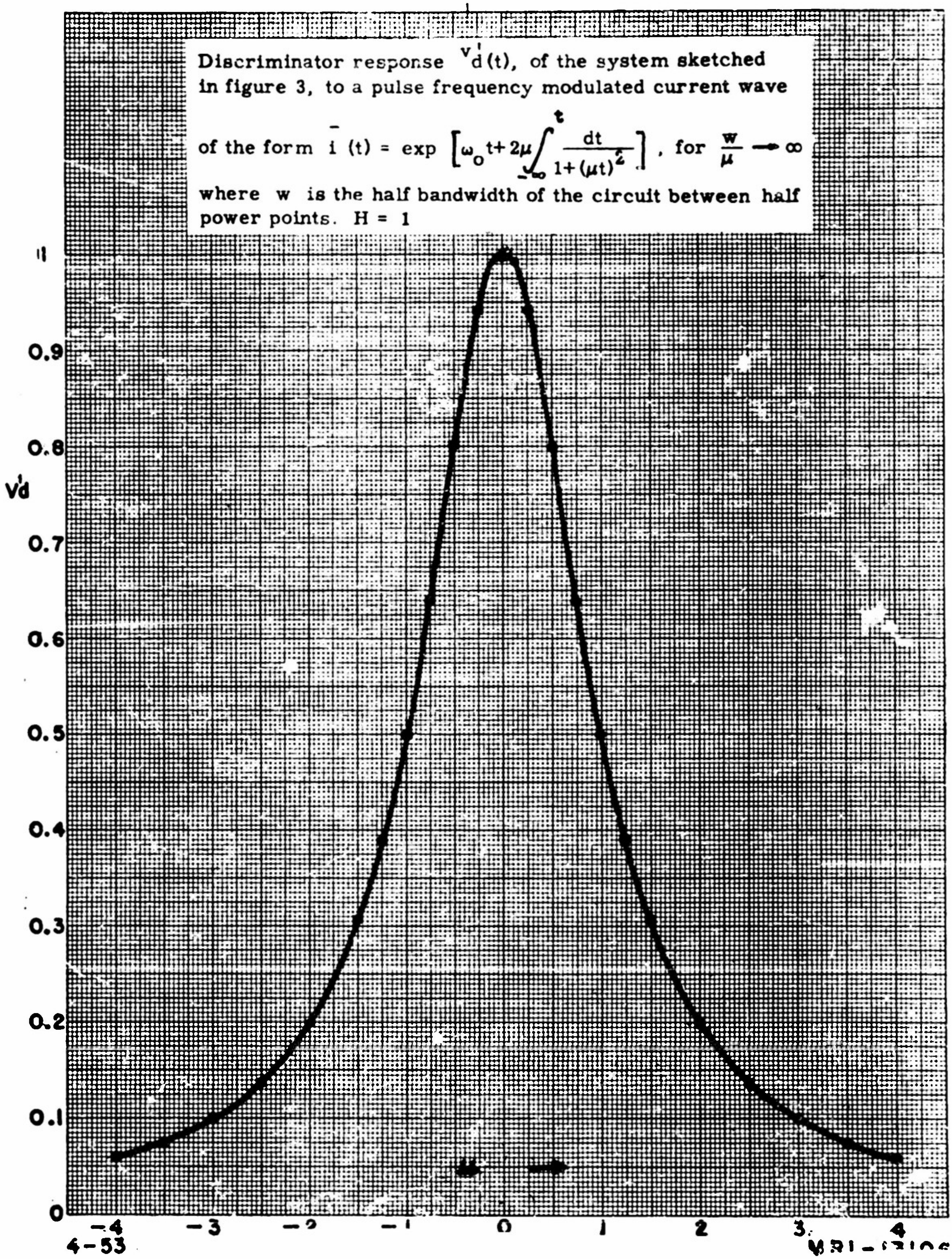
where

$$x = -\frac{w}{\mu}u \qquad\qquad y = -\frac{w}{\mu}$$

Conclusions

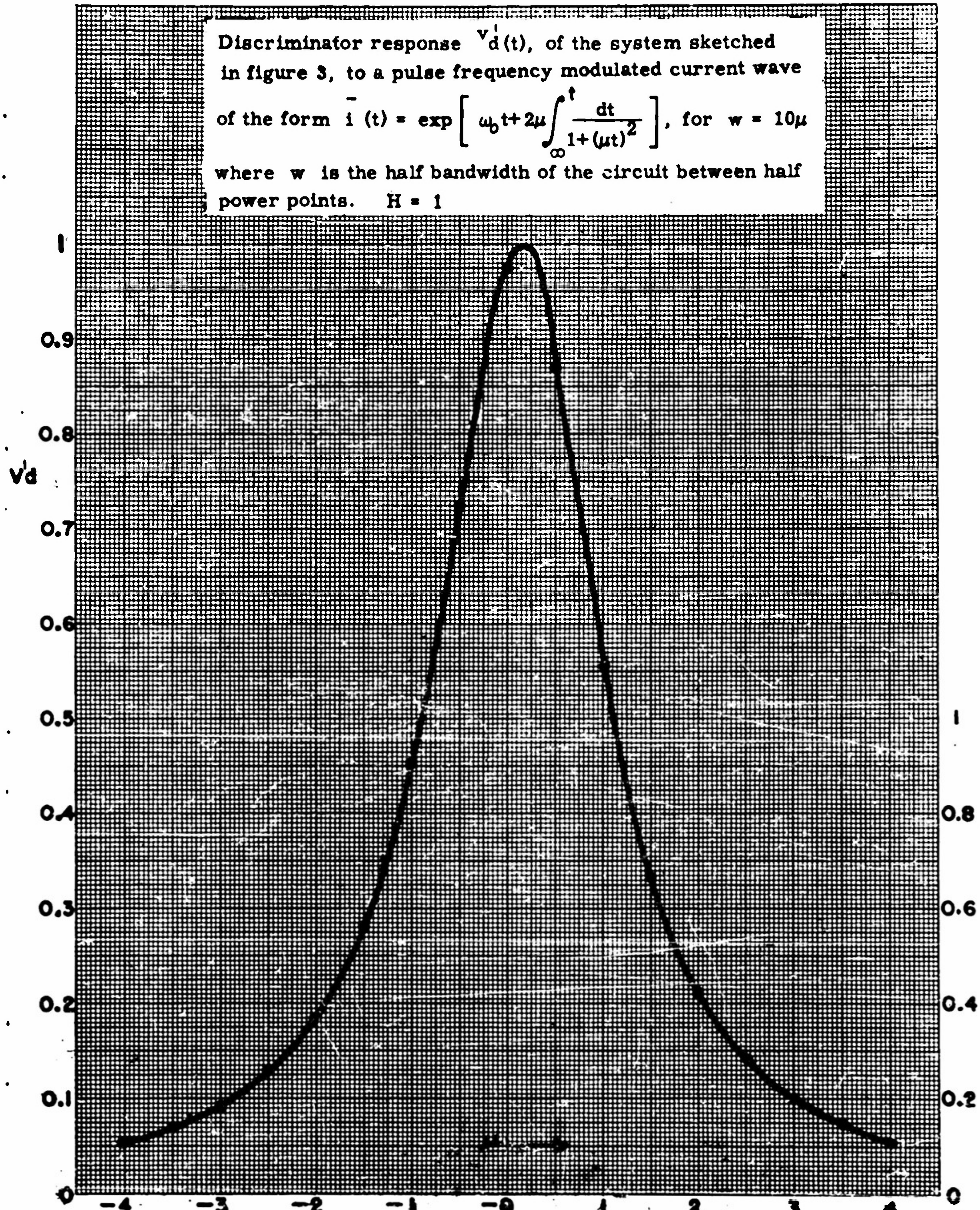
From the set of Figs. MRI-13106 to MRI-13115 it is apparent that the time dependence of v_d' is very similar to that of $S(t)$ if $w/\mu > 1$. This result is interesting since it shows that, even with a relatively narrow passband, the reproduction of the original signal is still good. To appreciate this fact consider a pulse of 10^{-6} sec. and assume for duration of the pulse the definition given in Section II. From Eq. (21) therefore we get $\mu = 6.10^6$, so that in order to get a good response it is necessary to assume $w > 6.10^6$, or the total bandwidth in cycles greater than approximately 2.10^6 c/sec. Figures MRI-13111 and MRI-13112 show, however, that it is not safe to operate in the neighborhood of $w/\mu = 1$, since for instance for $w/\mu = 0.4$ the response is very distorted.

It has to be emphasized that the numerical results obtained refer only to the case of $q = \Delta\,\omega_o/\mu = 1$. It would be interesting to carry on calculations for other values of q and eventually obtain formulae for non-integral values (<1) of q. However, it seems that the formulae become very cumbersome even for values of $q = 2$ or 3. It is also doubtful that it would be feasible to obtain formulae for $q \geqslant 3$ independent of ω_o.



Discriminator response $v_d'(t)$, of the system sketched in figure 3, to a pulse frequency modulated current wave of the form $\bar{i}(t) = \exp\left[\omega_0 t + 2\mu \int_{-\infty}^{t} \frac{dt}{1+(\mu t)^2}\right]$, for $\frac{w}{\mu} \to \infty$ where w is the half bandwidth of the circuit between half power points. $H = 1$

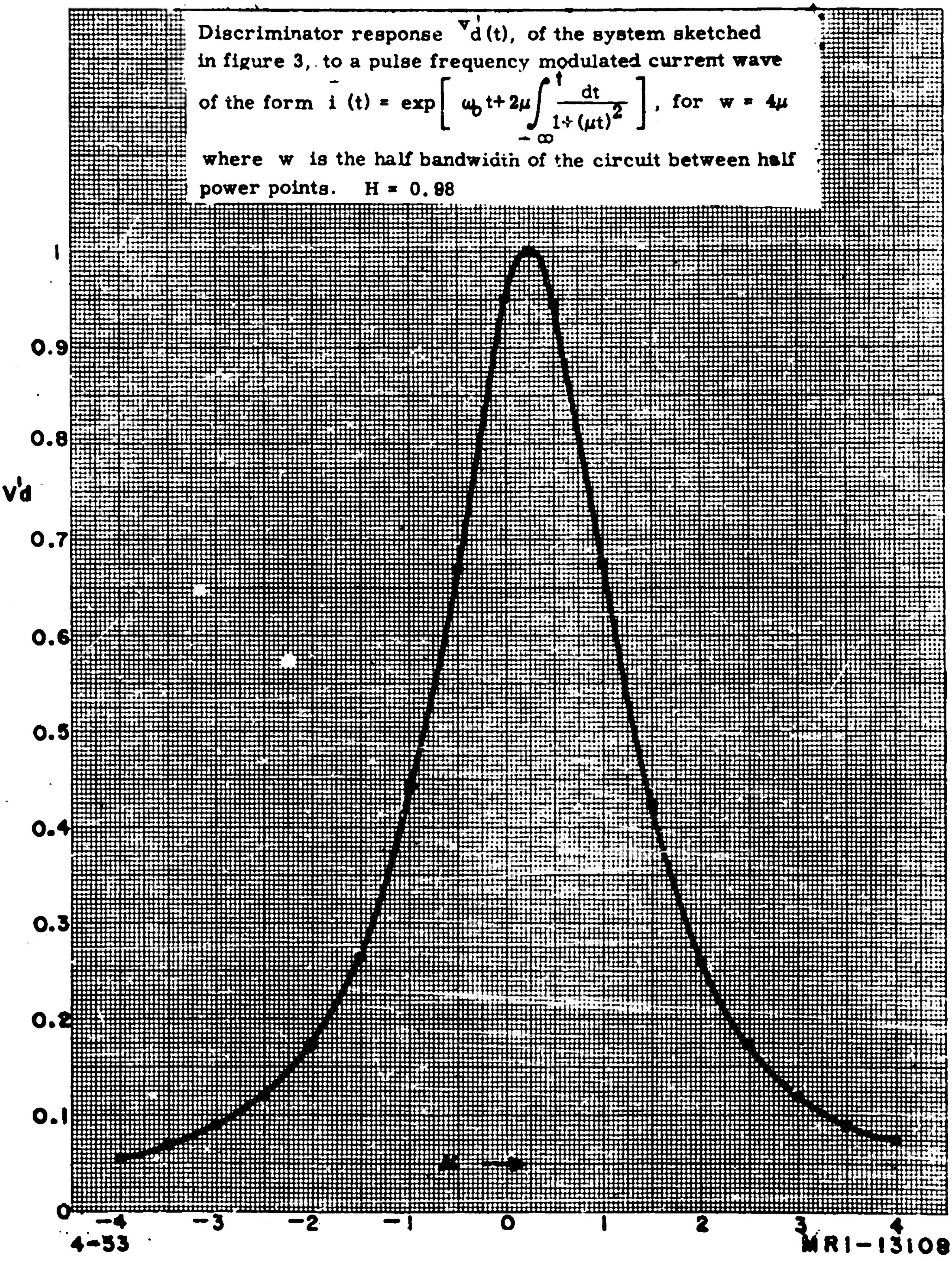
4-53



Discriminator response $v_d'(t)$, of the system sketched in figure 3, to a pulse frequency modulated current wave of the form $\bar{i}(t) = \exp\left[\omega_0 t + 2\mu \int_{\infty}^{t} \frac{dt}{1+(\mu t)^2}\right]$, for $w = 10\mu$ where w is the half bandwidth of the circuit between half power points. $H = 1$

4-53

MRI-13107

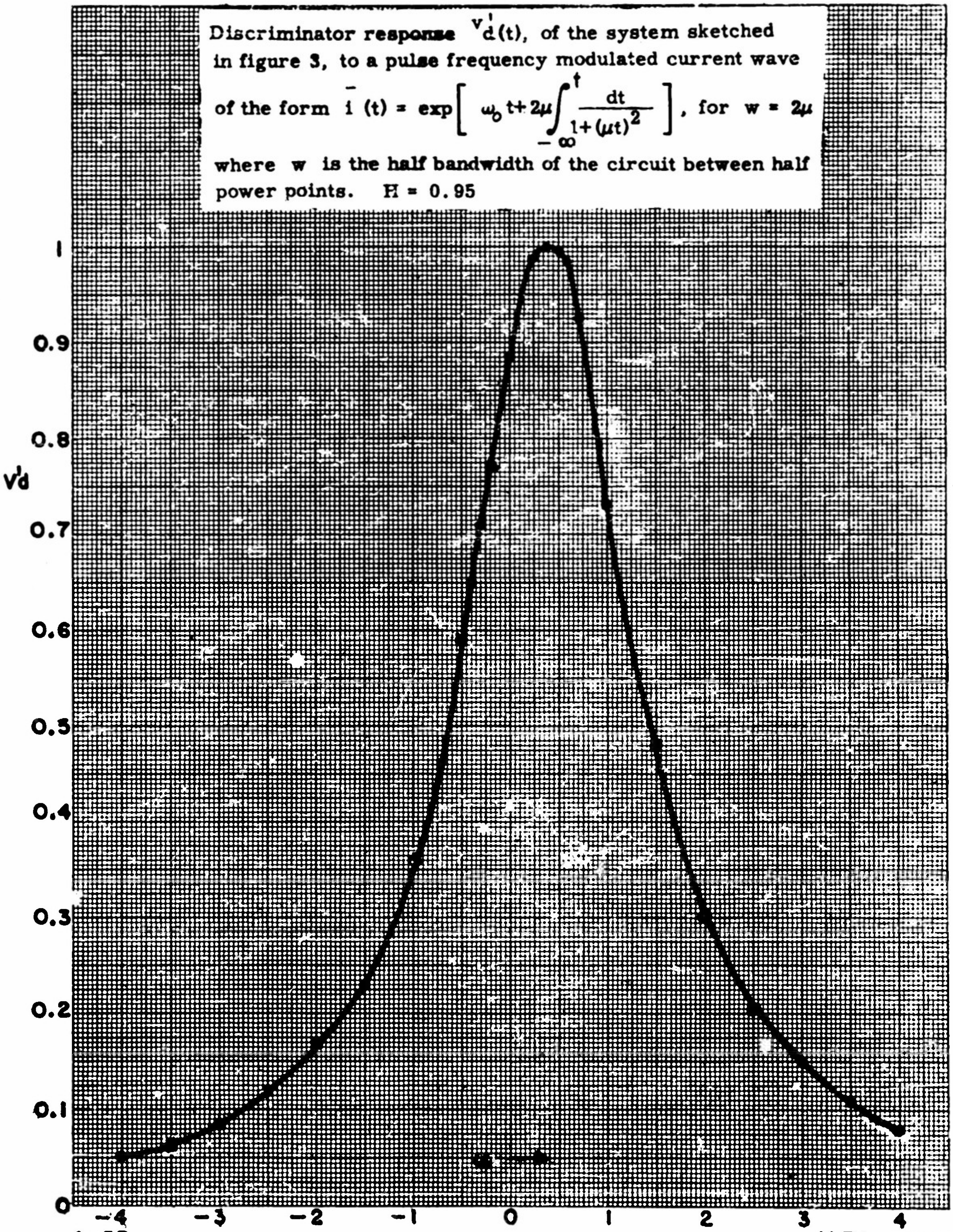


Discriminator response $v'_d(t)$, of the system sketched in figure 3, to a pulse frequency modulated current wave of the form $\bar{i}(t) = \exp\left[\omega_0 t + 2\mu \int_{-\infty}^{t} \frac{dt}{1+(\mu t)^2}\right]$, for $w = 4\mu$ where w is the half bandwidth of the circuit between half power points. $H = 0.98$

4-53

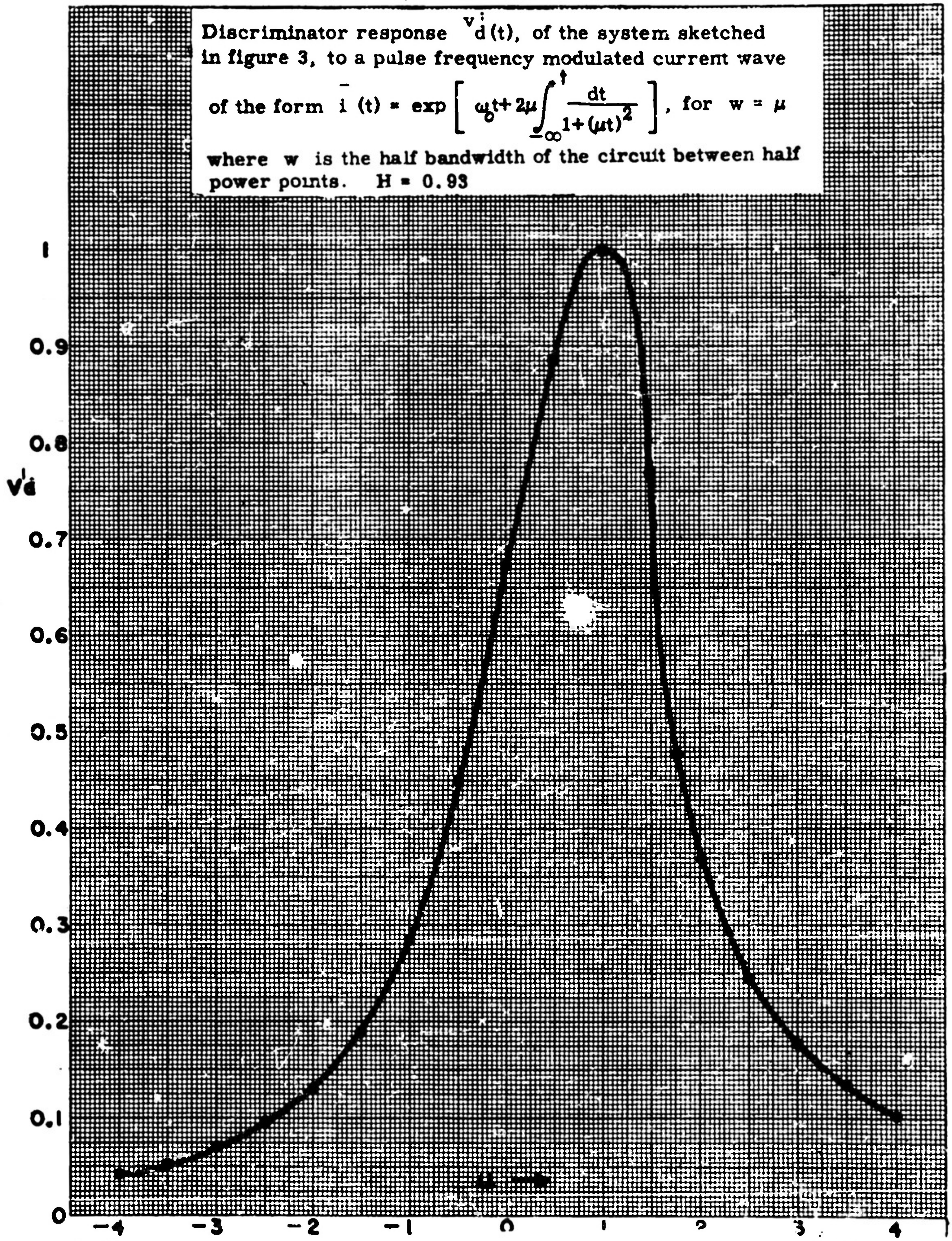
MRI-13108

Discriminator response $v'_d(t)$, of the system sketched in figure 3, to a pulse frequency modulated current wave of the form $\bar{i}(t) = \exp\left[\omega_0 t + 2\mu \int_{-\infty}^{t} \frac{dt}{1+(\mu t)^2}\right]$, for $w = 2\mu$ where w is the half bandwidth of the circuit between half power points. $H = 0.95$



4-53

MRI-13109

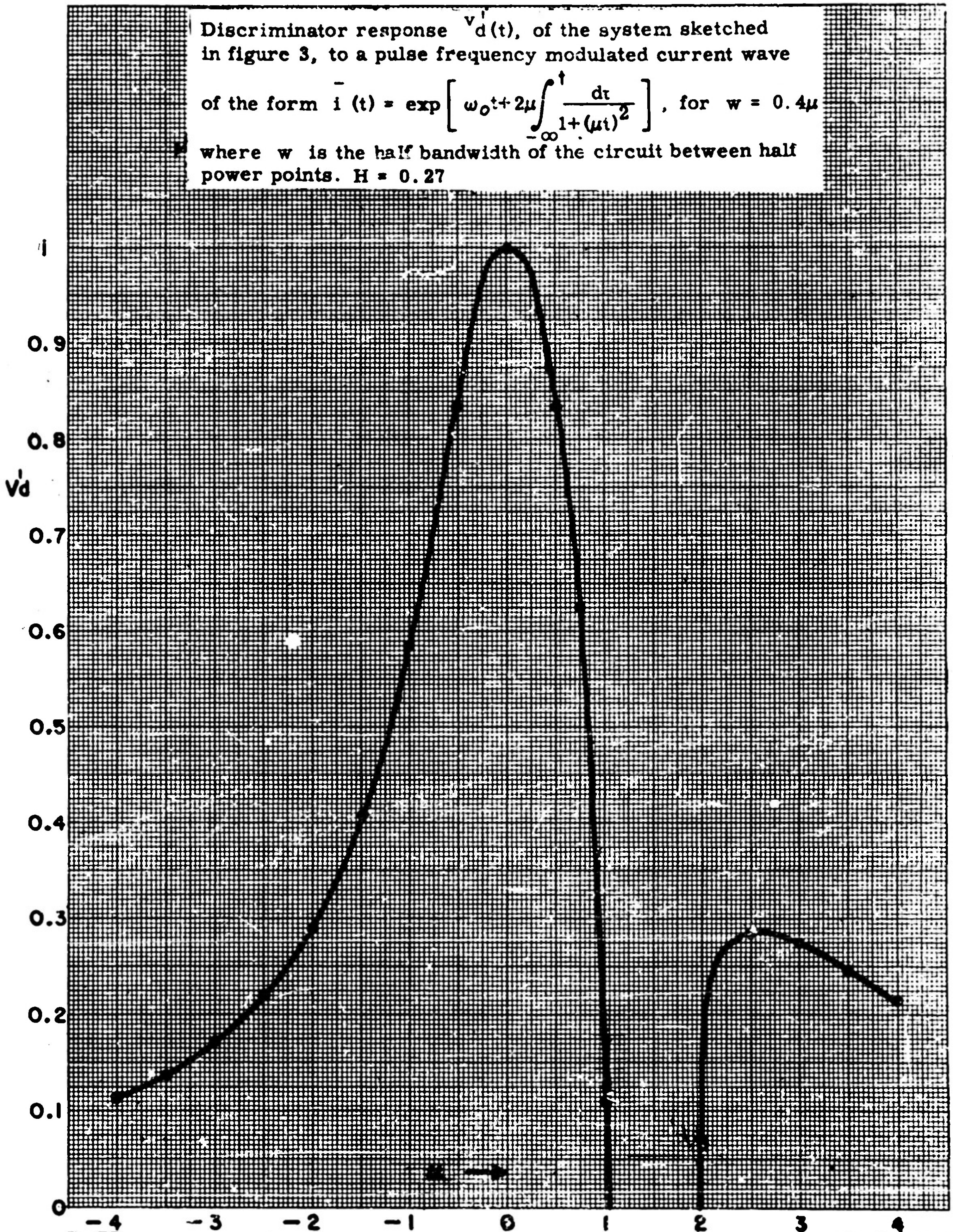


Discriminator response $v_d'(t)$, of the system sketched in figure 3, to a pulse frequency modulated current wave of the form $\bar{i}(t) = \exp\left[\omega_0 t + 2\mu \int_{-\infty}^{t} \frac{dt}{1+(\mu t)^2}\right]$, for $w = \mu$ where w is the half bandwidth of the circuit between half power points. $H = 0.93$

4-53

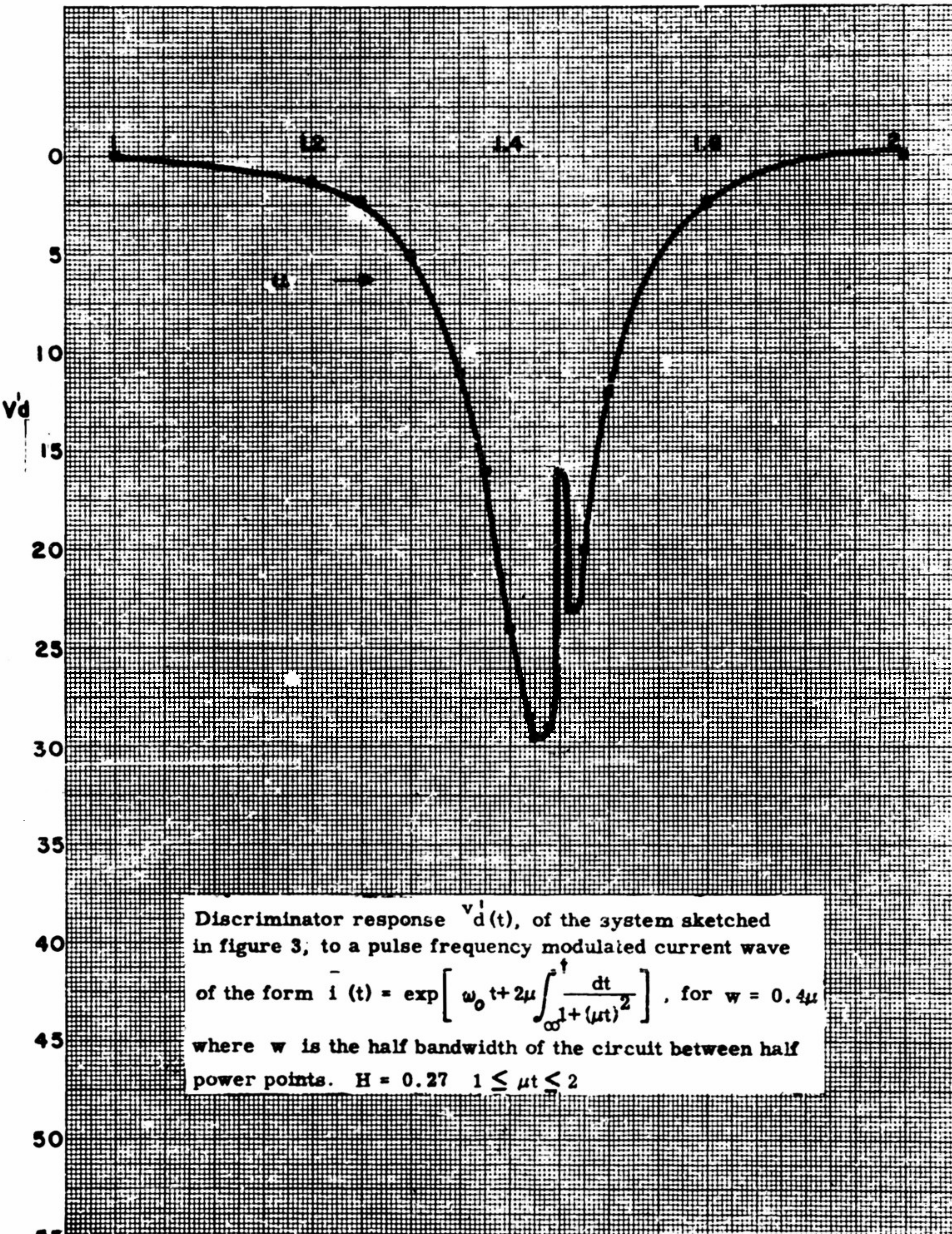
MRI-13110

Discriminator response $v'_d(t)$, of the system sketched in figure 3, to a pulse frequency modulated current wave of the form $\bar{i}(t) = \exp\left[\omega_o t + 2\mu \int_{-\infty}^{t} \frac{d\tau}{1+(\mu\tau)^2}\right]$, for $w = 0.4\mu$ where w is the half bandwidth of the circuit between half power points. H = 0.27

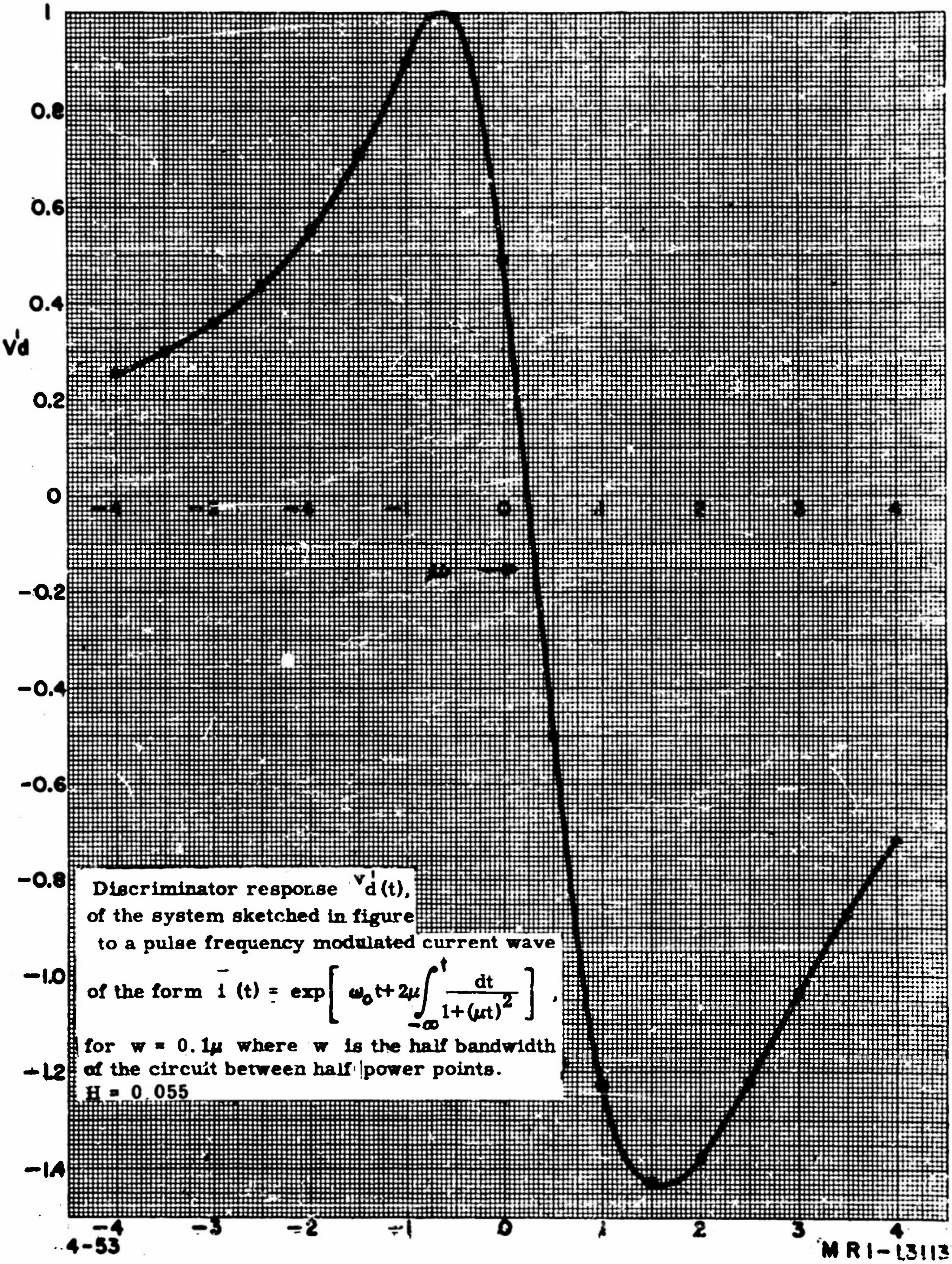


3-53

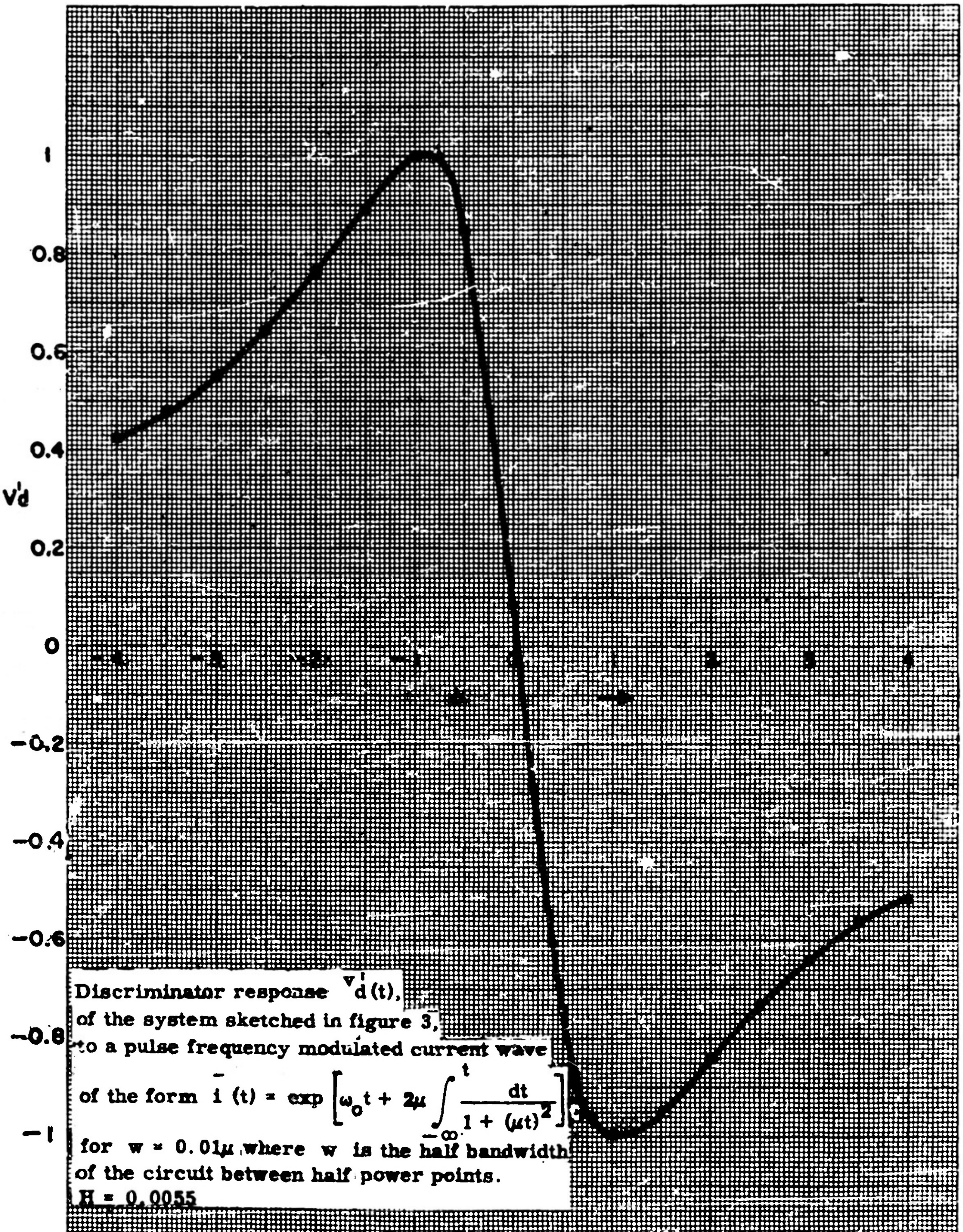
MRI-13111



Discriminator response $v'_d(t)$, of the system sketched in figure 3, to a pulse frequency modulated current wave of the form $\bar{i}(t) = \exp\left[\omega_0 t + 2\mu \int_{\infty}^{t} \frac{dt}{1+(\mu t)^2}\right]$, for $w = 0.4\mu$ where w is the half bandwidth of the circuit between half power points. $H = 0.27$ $\quad 1 \leq \mu t \leq 2$

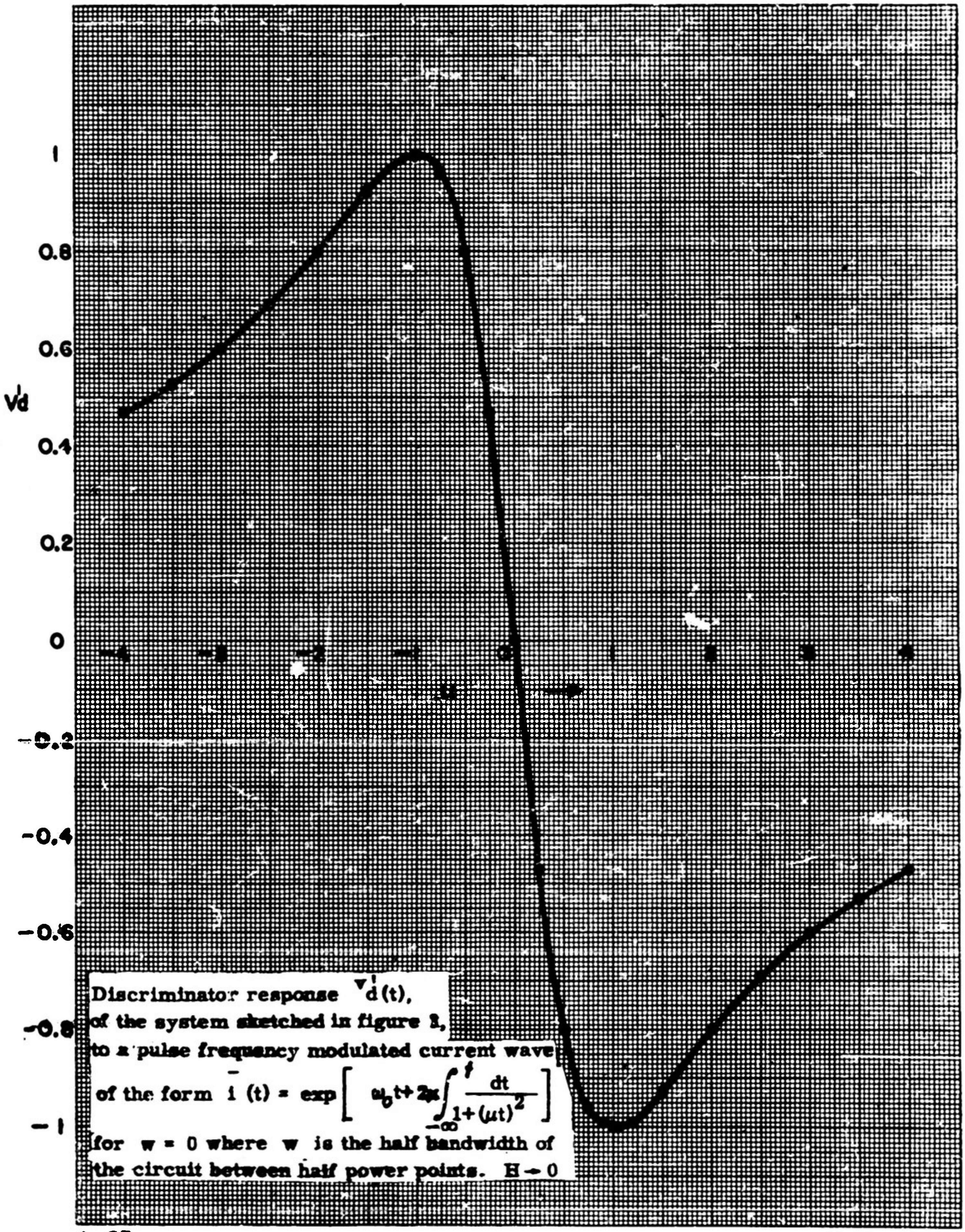


Discriminator response $v_d'(t)$, of the system sketched in figure to a pulse frequency modulated current wave of the form $\bar{i}(t) = \exp\left[\omega_0 t + 2\mu\int_{-\infty}^{t}\frac{dt}{1+(\mu t)^2}\right]$, for $w = 0.1\mu$ where w is the half bandwidth of the circuit between half power points. H = 0.055

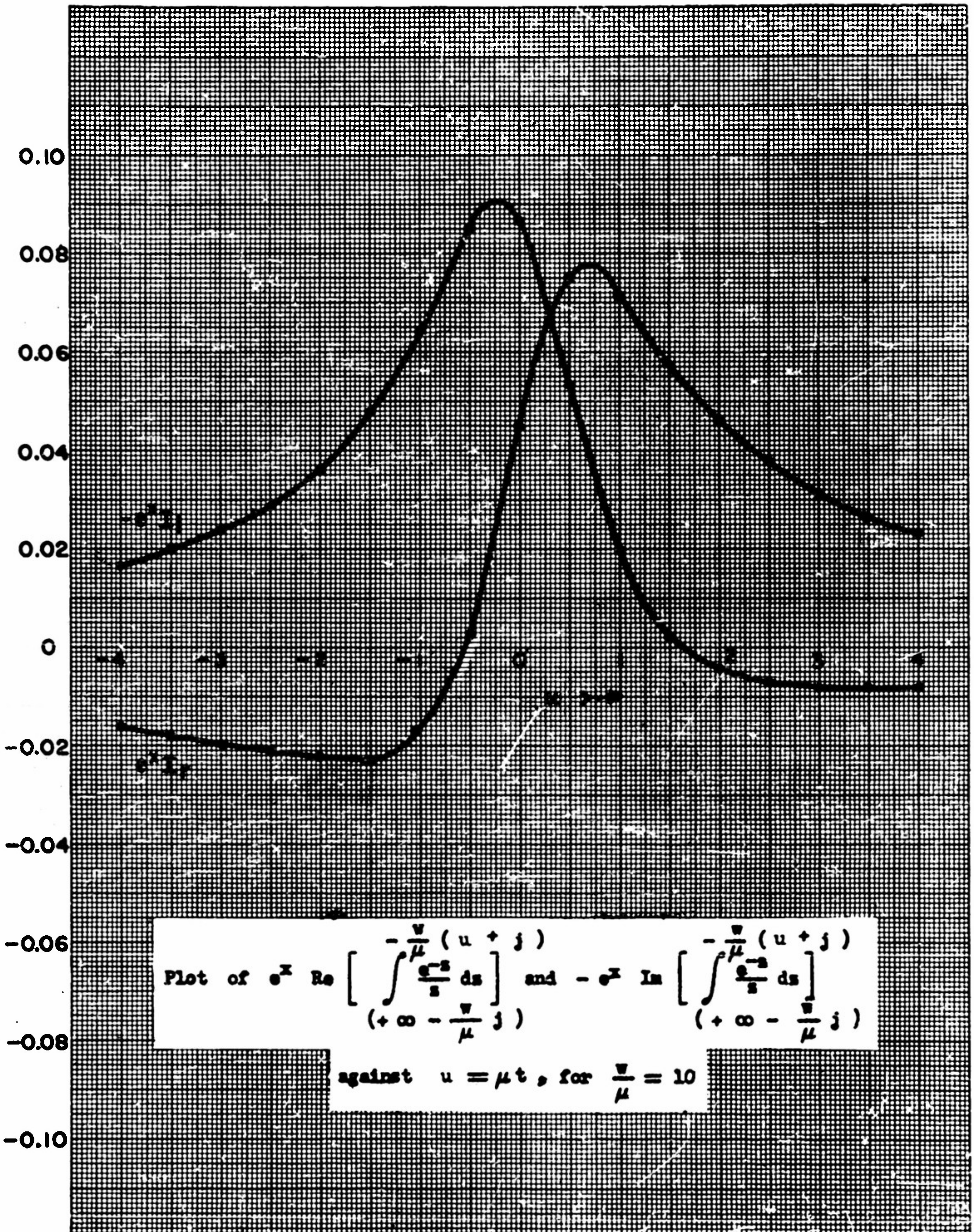


Discriminator response $v'_d(t)$, of the system sketched in figure 3, to a pulse frequency modulated current wave of the form $\bar{i}(t) = \exp\left[\omega_0 t + 2\mu \int_{-\infty}^{t} \frac{dt}{1 + (\mu t)^2}\right]$ for $w = 0.01\mu$ where w is the half bandwidth of the circuit between half power points.

H = 0.0055

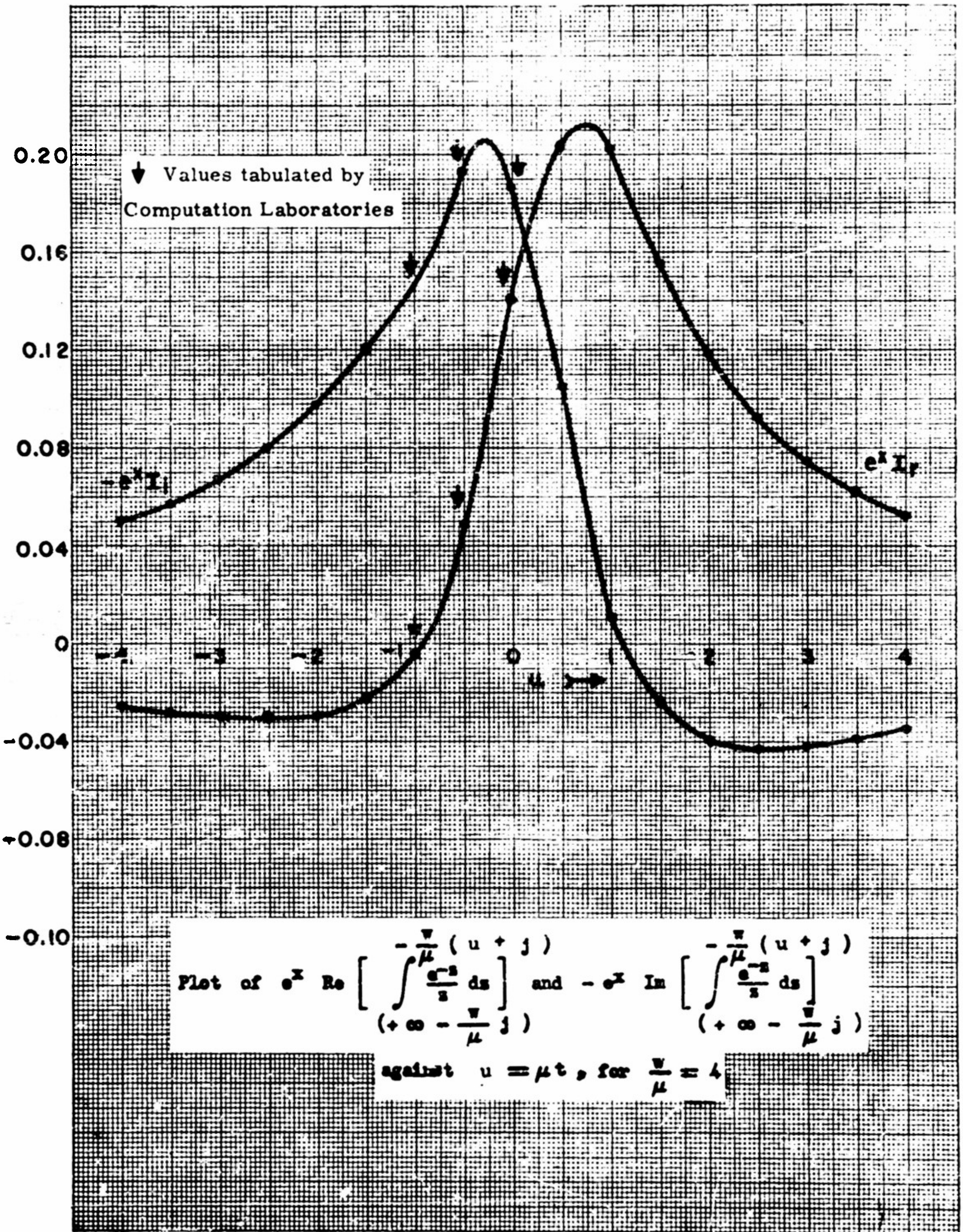


Discriminator response $v'_d(t)$, of the system sketched in figure 2, to a pulse frequency modulated current wave of the form $\bar{i}(t) = \exp\left[\omega_0 t + 2\mu \int_{-\infty}^{t} \frac{dt}{1+(\mu t)^2}\right]$ for w = 0 where w is the half bandwidth of the circuit between half power points. H → 0



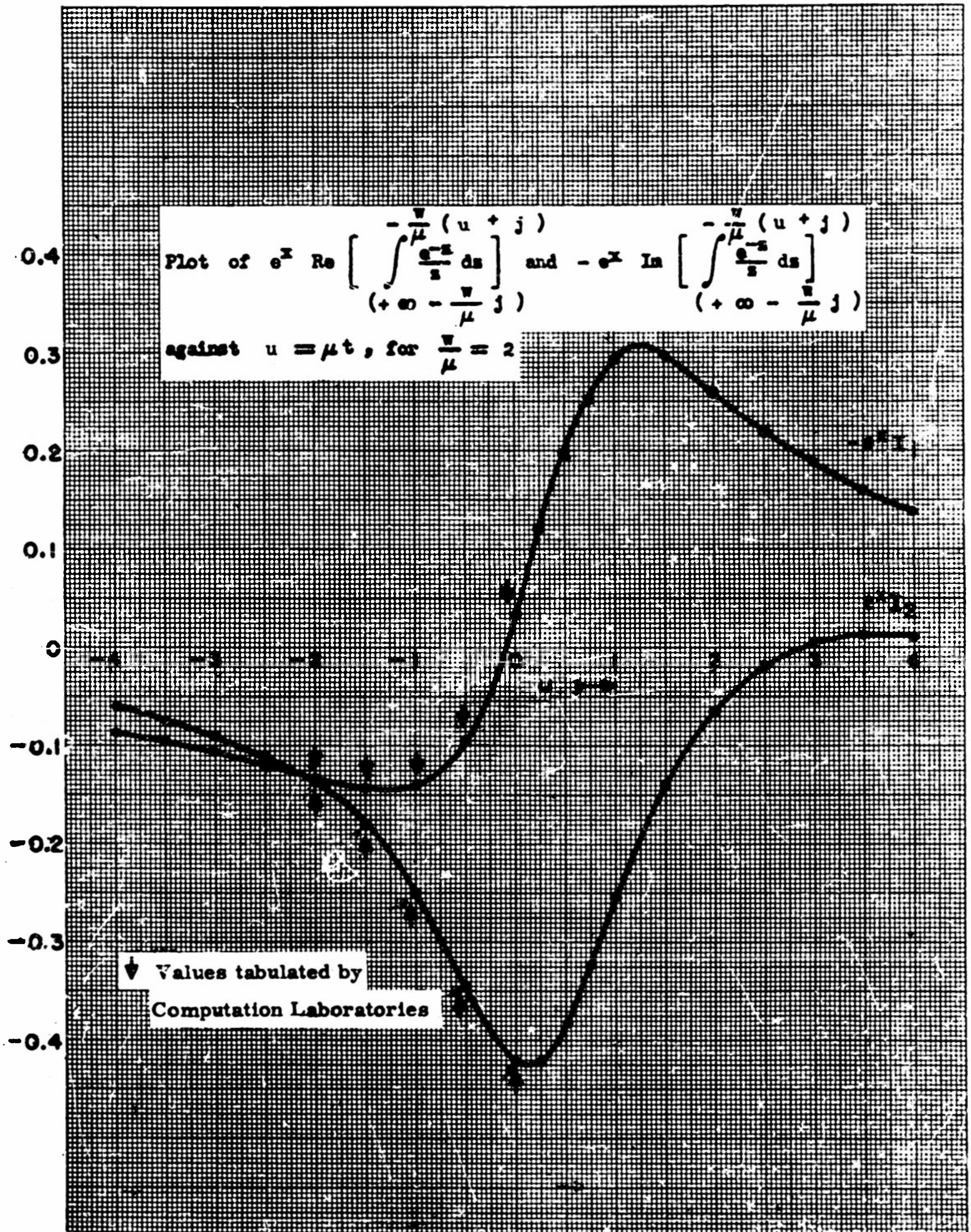
Plot of $e^x \operatorname{Re}\left[\int_{(+\infty - \frac{w}{\mu} j)}^{-\frac{w}{\mu}(u+j)} \frac{e^{-s}}{s}\,ds\right]$ and $-e^x \operatorname{Im}\left[\int_{(+\infty - \frac{w}{\mu} j)}^{-\frac{w}{\mu}(u+j)} \frac{e^{-s}}{s}\,ds\right]$

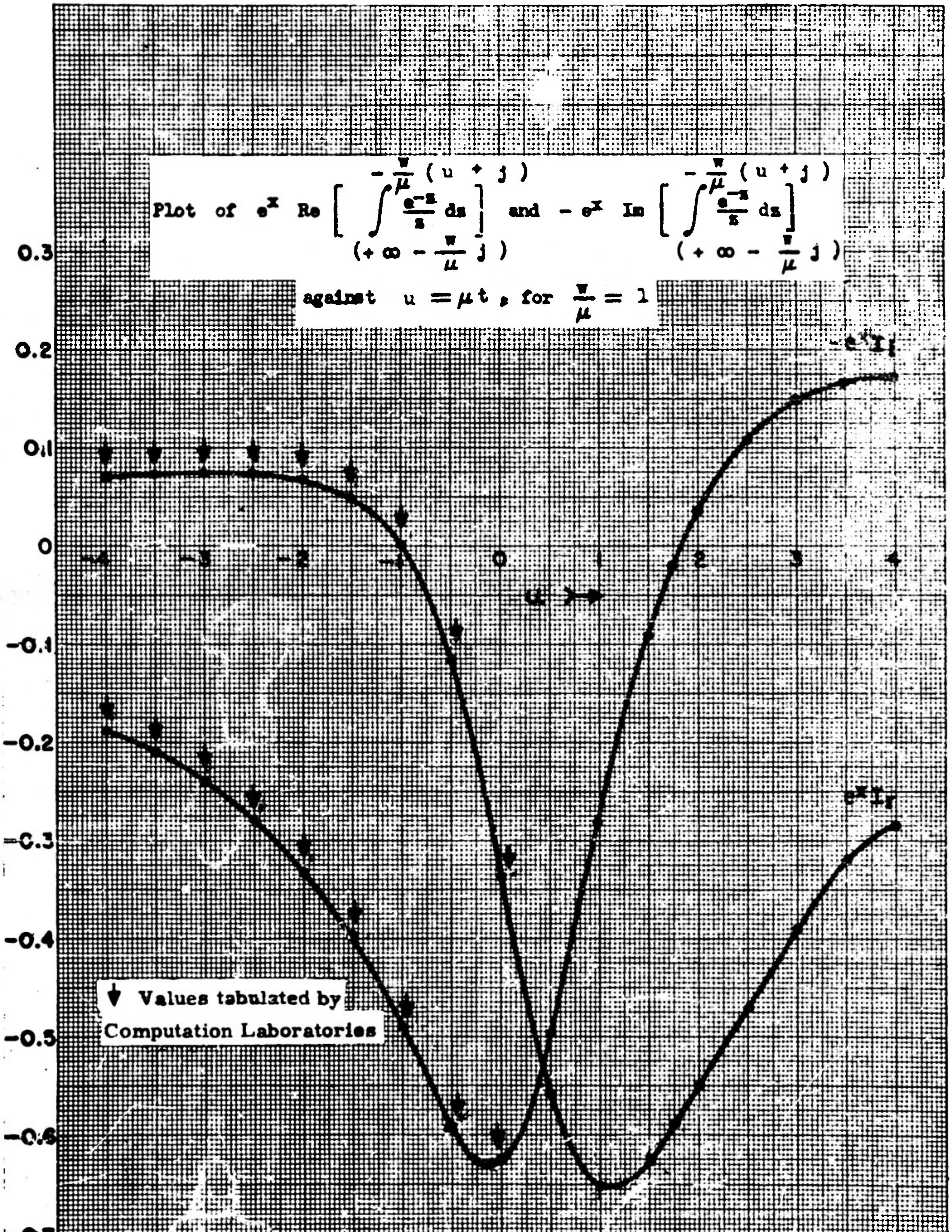
against $u = \mu t$, for $\frac{w}{\mu} = 10$



Plot of $e^x \; \mathrm{Re}\left[\int_{(+\infty - \frac{w}{\mu} j)}^{-\frac{w}{\mu}(u+j)} \frac{e^{-s}}{s}\, ds\right]$ and $-e^x \; \mathrm{Im}\left[\int_{(+\infty - \frac{w}{\mu} j)}^{-\frac{w}{\mu}(u+j)} \frac{e^{-s}}{s}\, ds\right]$

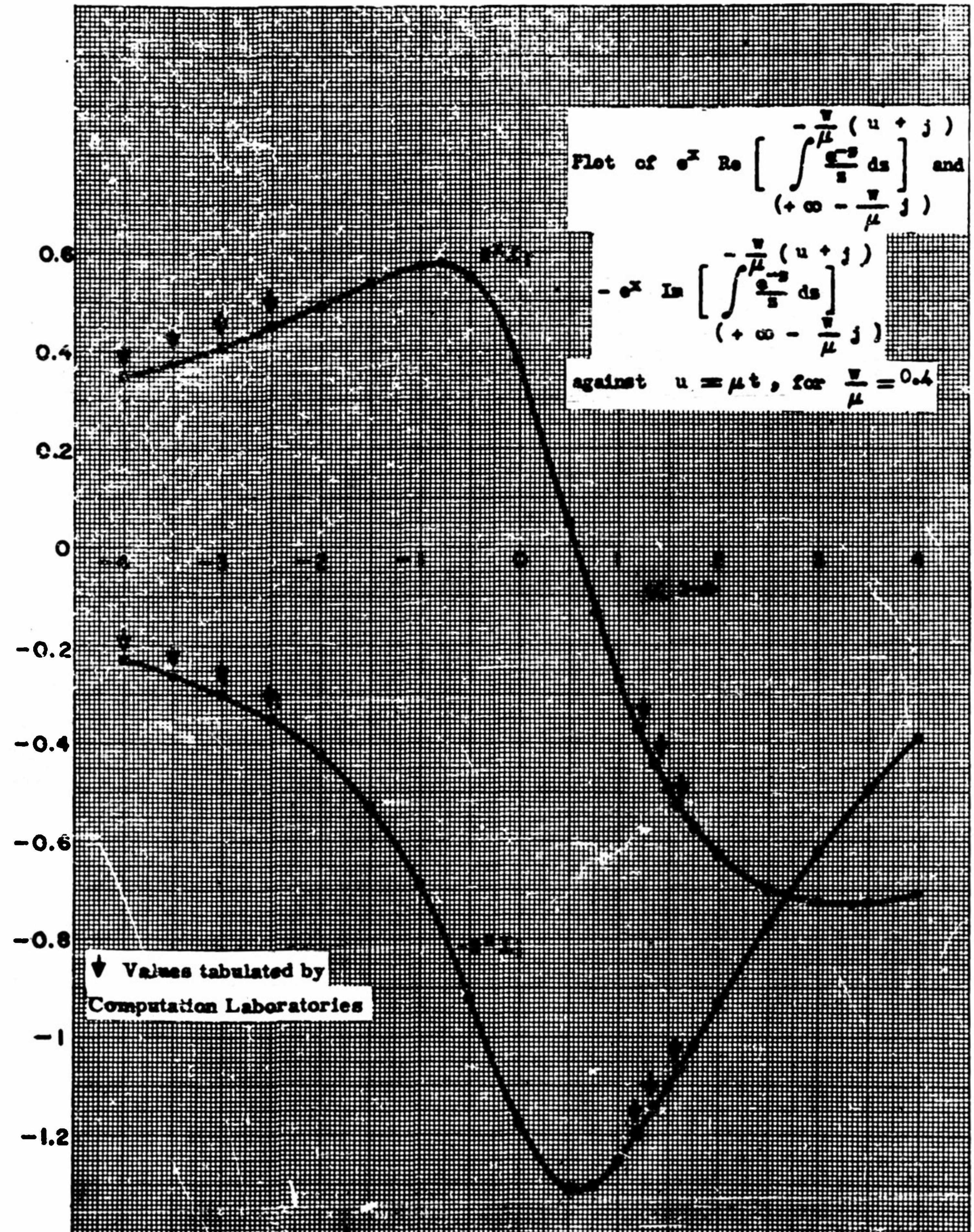
against $u = \mu t$, for $\frac{w}{\mu} = 4$

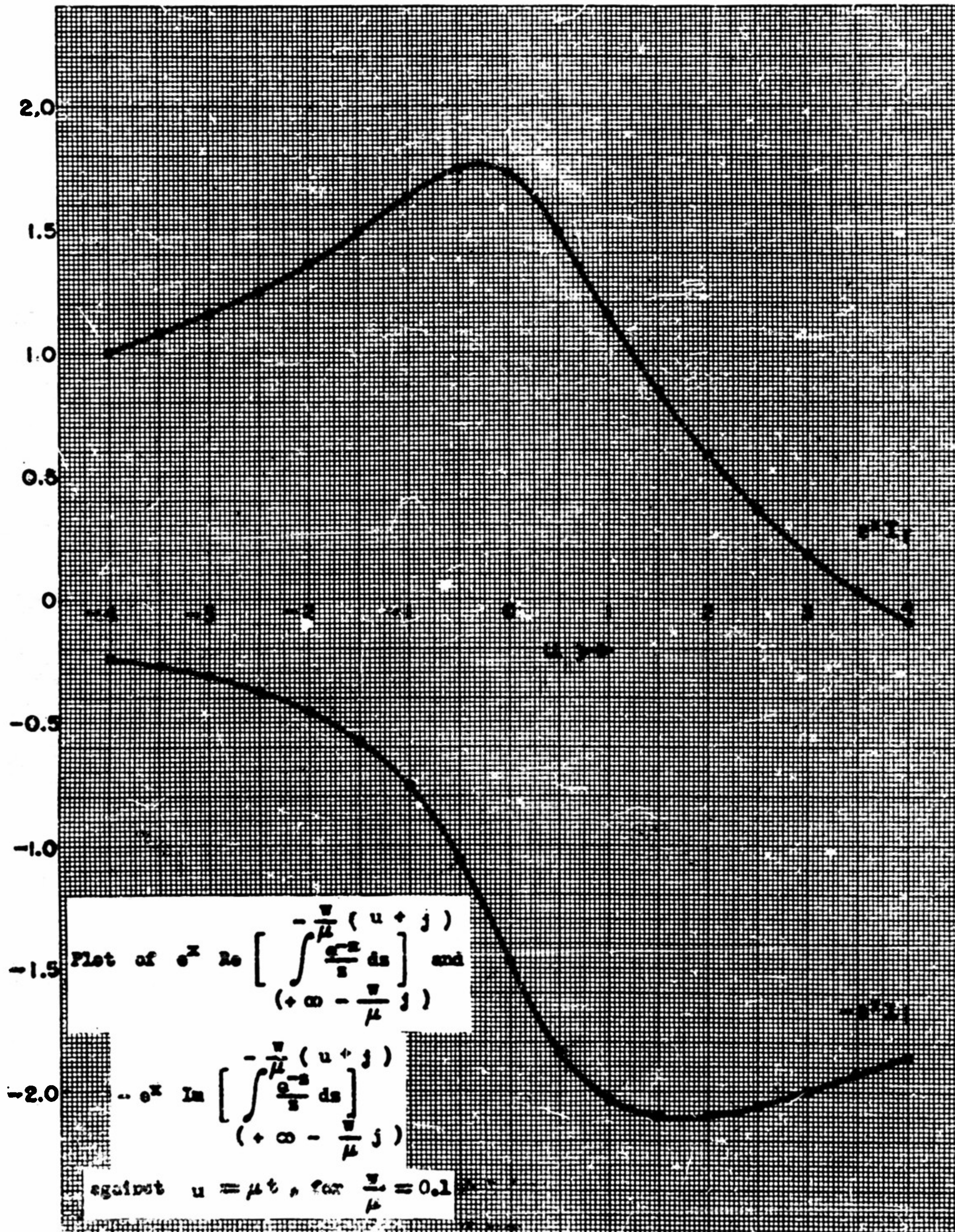

Plot of e^x Re $\left[\int_{(+\infty - \frac{w}{\mu} j)}^{-\frac{w}{\mu}(u+j)} \frac{e^{-z}}{z} dz\right]$ and $- e^x$ Im $\left[\int_{(+\infty - \frac{w}{\mu} j)}^{-\frac{w}{\mu}(u+j)} \frac{e^{-z}}{z} dz\right]$
against $u = \mu t$, for $\frac{w}{\mu} = 2$
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-4
-3
-2
-1
0
2
3
4
$-e^x I_1$
Values tabulated by
Computation Laboratories



4-53

MRI-13119


Plot of e^{x} Re $\left[\int_{(+\infty - \frac{w}{\mu} j)}^{-\frac{w}{\mu}(u + j)} \frac{e^{-s}}{s} ds\right]$ and
$- e^{x}$ Im $\left[\int_{(+\infty - \frac{w}{\mu} j)}^{-\frac{w}{\mu}(u + j)} \frac{e^{-s}}{s} ds\right]$
against $u = \mu t$, for $\frac{w}{\mu} = 0.4$
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-4
-3
-2
-1
0
1
2
3
4
Values tabulated by Computation Laboratories



Plot of $e^{x} \, \mathrm{Re}\left[\int_{(+\infty - \frac{w}{\mu} j)}^{-\frac{w}{\mu}(u + j)} \frac{e^{-s}}{s} \, ds\right]$ and

$-e^{x} \, \mathrm{Im}\left[\int_{(+\infty - \frac{w}{\mu} j)}^{-\frac{w}{\mu}(u + j)} \frac{e^{-s}}{s} \, ds\right]$

against $u = \mu t$, for $\frac{w}{\mu} = 0.1$